MINIMAL SURFACES THAT ATTAIN EQUALITY IN THE CHERN-OSSERMAN INEQUALITY

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ABSTRACT. In the previous paper, Takahasi and the authors generalized the theory of minimal surfaces in Euclidean n-space to that of surfaces with holomorphic Gauss map in certain class of non-compact symmetric spaces. It also includes the theory of constant mean curvature one surfaces in hyperbolic 3-space. Moreover, a Chern-Osserman type inequality for such surfaces was shown. Though its equality condition is not solved yet, the authors have noticed that the equality condition of the original Chern-Osserman inequality itself is not found in any literature except for the case $n = 3$, in spite of its importance. In this paper, a simple geometric condition for minimal surfaces that attains equality in the Chern-Osserman inequality is given. The authors hope it will be a useful reference for readers.

The total curvature $\mathrm{TC}(M)$ of any complete minimal surface M in $\mathbb{R}^n$ has a value in $2\pi\mathbb{Z}$ and satisfies the following inequality called the *Chern-Osserman inequality* [CO]:

$$\mathrm{TC}(M) \leq 2\pi(\chi_M - m), \tag{1}$$

where χ_M denotes the Euler number of M and m is the number of ends of M.

Then it is natural to ask which surfaces attain the equality of the inequality (1). In the case of $n = 3$, Jorge and Meeks [JM] gave a geometric proof of (1) and proved that the equality holds if and only if all of the ends are embedded. However, for general $n > 3$, the authors do not know any references on it. The purpose of this paper is to give the following geometric condition for attaining equality in (1) for general n.

Main Theorem. *A complete minimal surface in $\mathbb{R}^n$ attains equality in the Chern-Osserman inequality if and only if each end is asymptotic to a catenoid-type end or a planar end in some 3-dimensional subspace $\mathbb{R}^3$ in $\mathbb{R}^n$. In particular, all ends are embedded.*

For $n = 3$, according to Jorge-Meeks [JM] and Schoen [S], one can easily observe that embedded ends are all asymptotic to catenoids or planes (see Appendix). So our theorem generalizes the result in Jorge-Meeks. For general n (> 3), we remark that the embeddedness of ends is not a sufficient condition for the equality of (1). For example, an embedded holomorphic curve $f\colon \mathbb{C}\setminus\{0\} \to \mathbb{C}^2$ defined by $f(z) = (z, 1/z^2)$ (considered as a complete minimal surface in $\mathbb{R}^4$) has total curvature -6π. So it does not satisfy equality in (1).

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Preliminaries

We shall review the properties of minimal surfaces in $\mathbb{R}^n$ (cf. [L]). Let $f = (f_1, \dots, f_n) \colon M \to \mathbb{R}^n$ be a conformal minimal immersion of a Riemann surface M, where $n \geq 3$ is an integer. Then ∂f is a $\mathbb{C}^n$-valued holomorphic 1-form on M. We define the Gauss map $\nu \colon M \to \mathbb{CP}^{n-1}$ of f as

$$\nu := [\partial f] = \left[\frac{\partial f_1}{\partial z} : \frac{\partial f_2}{\partial z} : \cdots : \frac{\partial f_n}{\partial z}\right],$$

where z is a complex coordinate of M. Since f is conformal, we have

$$\langle \partial f, \partial f \rangle = \sum_{j=1}^{n} \left(\frac{\partial f_j}{\partial z}\right)^2 dz^2 = 0. \tag{2}$$

Thus, the Gauss map ν is valued in the complex quadric $\boldsymbol{Q}^{n-2} \subset \mathbb{CP}^{n-1}$.

We assume that f is complete and of finite total curvature. Under this assumption, the following properties are well-known:

- M is biholomorphic to a compact Riemann surface $\overline{M}$ punctured at finitely many points $\{p_1, \dots, p_m\}$. Each point p_j is called an *end*.
- The Gauss map ν can be extended holomorphically on $\overline{M}$, and the total curvature is given by $-2\pi d$ where d is the homology degree of $\nu(\overline{M})$ in $\mathbb{CP}^{n-1}$.
- For each end p_j, there exists a local complex coordinate z on $\overline{M}$ centered at p_j such that the first fundamental form ds^2 is written as

  $$ds^2 = |z|^{2\mu_j} dz\, d\bar{z} \qquad (\mu_j \leq -2).$$

  We call μ_j the *order* of the metric ds^2 at the end p_j and denote by $\operatorname{ord}_{p_j} ds^2 = \mu_j$. Since $ds^2 = 2\langle \partial f, \overline{\partial} f \rangle$, μ_j coincides with the order of ∂f at the end p_j.

Definition 1. An end p_j of $f \colon M = \overline{M} \setminus \{p_1, \dots, p_m\} \to \mathbb{R}^n$ is said to be asymptotic to a *catenoid-type* (resp. *planar*) end if there exists a piece of the catenoid (resp. the plane)

$$f_0 \colon \{|z - p_j| < \varepsilon\} \to \mathbb{R}^3 \subset \mathbb{R}^n$$

which is complete at p_j such that $|f(z) - f_0(z)| = O(|z - p_j|)$, that is,

$$\frac{|f(z) - f_0(z)|}{|z - p_j|}$$

is bounded on $\{|z - p_j| < \varepsilon\}$ for sufficiently small $\varepsilon > 0$.

Proof of the Main Theorem

The Chern-Osserman inequality follows from the fact $\operatorname{ord}_{p_j} ds^2 \leq -2$ at each end p_j. Moreover, equality holds if and only if $\operatorname{ord}_{p_j} ds^2 = -2$ (see [L, pp. 135–136], for example). Thus the Main Theorem immediately follows from the following Lemma.

Lemma 2. *Let $f\colon \Delta^* \to \mathbb{R}^n$ be a conformal minimal immersion of a punctured disc $\Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < 1\}$ into $\mathbb{R}^n$ which is complete at the origin 0. Then* $\operatorname{ord}_0 ds^2 = -2$ *holds if and only if the end 0 is asymptotic to a catenoid-type end or a planar end in $\mathbb{R}^3 (\subset \mathbb{R}^n)$. In particular, it is an embedded end.*

Proof. Suppose that $\operatorname{ord}_0 ds^2 = -2$. It implies that the Laurent expansion of ∂f is given by

$$(3) \qquad \partial f = \left(\frac{1}{z^2}\boldsymbol{a}_{-2} + \frac{1}{z}\boldsymbol{a}_{-1} + \cdots\right) dz, \quad \boldsymbol{a}_{-2} \in \mathbb{C}^n \setminus \{0\},\ \boldsymbol{a}_{-1} \in \mathbb{R}^n$$

because the residue of ∂f must be real. Moreover, it follows from (2) that

$$\langle \boldsymbol{a}_{-2}, \boldsymbol{a}_{-2} \rangle = 0, \quad \text{and} \quad \langle \boldsymbol{a}_{-2}, \boldsymbol{a}_{-1} \rangle = 0.$$

Therefore we have

$$\begin{aligned} |\operatorname{Re} \boldsymbol{a}_{-2}| &= |\operatorname{Im} \boldsymbol{a}_{-2}|, & \langle \operatorname{Re} \boldsymbol{a}_{-2}, \operatorname{Im} \boldsymbol{a}_{-2} \rangle &= 0, \\ \langle \operatorname{Re} \boldsymbol{a}_{-2}, \boldsymbol{a}_{-1} \rangle &= 0, & \langle \operatorname{Im} \boldsymbol{a}_{-2}, \boldsymbol{a}_{-1} \rangle &= 0. \end{aligned}$$

Hence we can choose an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of $\mathbb{R}^n$ so that

$$\operatorname{Re} \boldsymbol{a}_{-2} = a\,\boldsymbol{e}_1, \quad \operatorname{Im} \boldsymbol{a}_{-2} = a\,\boldsymbol{e}_2, \quad \boldsymbol{a}_{-1} = b\,\boldsymbol{e}_3$$

for some real constants $a(\neq 0)$, b. With respect to this basis, we have

$$\partial f = \left(\frac{a}{z^2}(\boldsymbol{e}_1 + i\boldsymbol{e}_2) + \frac{b}{z}\boldsymbol{e}_3 + \cdots\right) dz, \quad a, b \in \mathbb{R}, (a \neq 0)$$

Then using the polar coordinate $z = re^{i\theta}$, we have

$$(4) \qquad f(z) = 2\int_{z_0}^{z} \partial f = -\frac{2a\cos\theta}{r}\boldsymbol{e}_1 - \frac{2a\sin\theta}{r}\boldsymbol{e}_2 + 2b\log r\,\boldsymbol{e}_3 + O(r),$$

where z_0 is a base point. Here, we have dropped the constant terms in $f(z)$ by a suitable parallel translation. By Definition 1, the formula (4) implies that the surface $f(\Delta^*)$ is asymptotic to the catenoid (resp. the plane) for the sufficiently small r if $b \neq 0$ (resp. if $b = 0$).

Conversely, suppose that $\operatorname{ord}_0 ds^2 \neq -2$. It implies that $\operatorname{ord}_0 ds^2 = -k$ $(k \geq 3)$ and

$$(5) \quad \partial f = \left(\frac{1}{z^k}\boldsymbol{a}_{-k} + \cdots + \frac{1}{z}\boldsymbol{a}_{-1} + \cdots\right) dz, \quad \boldsymbol{a}_{-k} \neq 0 \in \mathbb{C}^n,\ \boldsymbol{a}_{-1} \in \mathbb{R}^n.$$

It is obvious that the end is asymptotic to neither a catenoid-type end nor a planar end.

From now on, we shall prove that an end is embedded if it is asymptotic to a catenoid-type end or a planar end. Assume that the end is not embedded. Then there exist two sequences $\{z_j\}$, $\{z'_j\}$ convergent to 0 such that $f(z_j) = f(z'_j)$ for all j. Then by (4), there exists a positive constant C such that

$$\left|\frac{\cos\theta_j}{r_j} - \frac{\cos\theta'_j}{r'_j}\right| \leq C|r_j - r'_j|, \qquad \left|\frac{\sin\theta_j}{r_j} - \frac{\sin\theta'_j}{r'_j}\right| \leq C|r_j - r'_j|,$$

where $z_j = r_j e^{i\theta_j}$ and $z'_j = r'_j e^{i\theta'_j}$ $(j = 1, 2, \dots)$. With these estimates, we have

$$\begin{aligned}\left(\frac{1}{r_j} - \frac{1}{r'_j}\right)^2 &\leq \frac{1}{{r_j}^2} + \frac{1}{{r'_j}^2} - \frac{2}{r_j r'_j}\cos(\theta_j - \theta'_j) \\ &= \left|\frac{\cos\theta_j}{r_j} - \frac{\cos\theta'_j}{r'_j}\right|^2 + \left|\frac{\sin\theta_j}{r_j} - \frac{\sin\theta'_j}{r'_j}\right|^2 \\ &\leq 2C^2 |r_j - r'_j|^2,\end{aligned}$$

and then,

$$\frac{1}{(r_j r'_j)^2} \leq 2C^2 \tag{6}$$

holds. However the left hand side of (6) diverges to $+\infty$ as $j \to \infty$. This is a contradiction. □

Besides the Chern-Osserman inequality (1), the following inequalities for fully immersed complete minimal surfaces are known. (We say that the immersion f is *full* if the image $f(M)$ is not contained in any hyperplanes of $\mathbb{R}^n$.)

Gackstatter [G] proved that

$$\mathrm{TC}(M) \leq (2\chi_M + m - 1 - n)\pi.$$

On the other hand, Ejiri [E] proved the inequality

$$\mathrm{TC}(M) \leq (\chi_M + m - 2n + 2l)\pi \tag{7}$$

if its Gauss image $\nu(M)$ is contained in an $(n-1-l)$-dimensional subspace of $\mathbb{CP}^{n-1}$.

Here, we shall give a new example of complete minimal surfaces which satisfies the equality both in the Chern-Osserman equality (1) and in the Ejiri inequality (7).

Example (Generalized Jorge-Meeks' surface)**.** For $j = 0, 1, \dots, m-1$, we put

$$g_j(z) = \frac{z^j(1 - z^{2m-2j})}{(z^{m+1} - 1)^2}, \quad h_j(z) = \frac{iz^j(1 + z^{2m-2j})}{(z^{m+1} - 1)^2},$$

and define a complete conformal minimal immersion by

$$f_m := \mathrm{Re}\int_{z_0}^{z} \left(g_0, h_0, g_1, h_1, \dots, g_{m-1}, h_{m-1}, \frac{2\sqrt{m}z^m}{(z^{m+1}-1)^2}\right) dz. \tag{8}$$

Then by similar computations as in [JM], the integrand of (8) has real residue at each pole, and then, f_m gives a conformal minimal immersion

$$f_m \colon M = \bigl(\mathbb{C} \cup \{\infty\}\bigr) \setminus \{z\,;\, z^{m+1} = 1\} \longrightarrow \mathbb{R}^{2m+1}.$$

Obviously, the genus of M is zero, the number of ends is $m+1$, and $f_m \colon M \to \mathbb{R}^{2m+1}$ is full.

Since the degree of the Gauss map of f_m is $2m$, the total curvature $\mathrm{TC}(M)$ is equal to $-4m\pi$. Therefore it attains the equality in the Chern-Osserman inequality.

On the other hand, it is easy to see that f_m has non-degenerate Gauss map, that is, $l = 0$ in (7). Then the right hand side of (7) is $-4m\pi$. Hence the equality in (7) holds.

Appendix: Embedded ends in $\mathbb{R}^3$

For the case $n = 3$, embeddedness of the end 0 in Lemma 2 implies $\operatorname{ord}_0 ds^2 = -2$, and consequently the end is asymptotic to a catenoid-type end or a planer end ([JM, Theorem 4] or [S, Proposition 1]). Here we shall give a simple proof of this fact, which is a mixture of Jorge-Meeks' and Schoen's. The authors hope that it will be helpful to readers. The crucial point of the Jorge-Meeks' proof is to show that the intersection of the end and the sphere of radius r centered at the origin converges to a finite covering of a great sphere as $r \to \infty$. According to Schoen [S], we prove it via the Weierstrass representation directly.

Consider the Laurent expansion as (5) for $k \geq 2$. Without loss of generality, we may set $\boldsymbol{a}_{-k} = (a, ia, 0)$ $(a \in \mathbb{R} \setminus \{0\})$ because of (2). Integrating this, we have

$$f(re^{i\theta}) = \frac{1}{r^{k-1}} \left[2a\bigl(\cos(k-1)\theta, \sin(k-1)\theta, 0\bigr) + o(1)\right],$$

where $o(1)$ means a term tending to 0 as $r \to 0$. Let S^2_R be the sphere in $\mathbb{R}^3$ with radius R centered at the origin and consider the intersection of the surface and S^2_R:

$$E_R := \frac{1}{R}\left(S^2_R \cap f(\Delta^*)\right) \subset S^2_1,$$

which is normalized as a subset of the unit sphere.

Here, $f \in S^2_R$ if and only if

$$R^2 = f_1^2 + f_2^2 + f_3^2 = \frac{1}{r^{2k-2}}(4a^2 + o(1))$$

holds. Then $r \to 0$ as $R \to \infty$ when $f(re^{i\theta}) \in S^2_R$ because $k \geq 2$. In particular, $\lim_{r\to\infty} R^2 r^{2k-2} = 4a^2$ holds.

Then under the condition $f(z) \in S^2_R$,

$$\lim_{R\to\infty} \frac{1}{R} f(re^{i\theta}) = \bigl(\cos(k-1)\theta, \sin(k-1)\theta, 0\bigr)$$

holds. This implies that, for sufficiently large R, E_R is a closed curve in a neighborhood of the equator of S^2_1 with rotation index $|k-1|$, which is embedded if and only if $k = 2$. $\square$

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References

[CO] S. Chern and R. Osserman, *Complete minimal surface in Euclidean n-space,* J. Analyse Math., **19** (1967) 15–34.

[E] N. Ejiri, *Degenerate minimal surfaces of finite total curvature in R^N*, Kobe J. Math., **14** (1997), 11–22.

[G] F. Gackstatter, *Über die Dimension einer Minimalfläche und zur Ungleichung von St. Cohn-Vossen,* Arch. Rational Mech. Anal., **61** (1976), 141–152.

[JM] L. P. M. Jorge and W. H. Meeks III, *The topology of complete minimal surfaces of finite total curvature,* Topology, **22** (1983), 203–221.

[KTUY] M. Kokubu, M. Takahashi, M. Umehara and K. Yamada, *An analogue of minimal surface theory in* $SL(\boldsymbol{n}, \mathbb{C})/SU(\boldsymbol{n})$, Preprint.

[L] H. B. Lawson, LECTURES ON MINIMAL SUBMANIFOLDS (VOLUME 1), Publish or Perish Inc., 1980.

[S] R. Schoen, *Uniqueness, symmetry and embeddedness of minimal surfaces,* J. Differential Geometry, **18** (1983), 791–809.

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